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ABSTRACT

This study investigated students' generalizations of numeric situations through the use of a whole-class teaching experiment in which students were provided problem situations in order to: (1) trace students' understanding of recursive and explicit reasoning; and (2) describe the sociomathematical norms and classroom practices that were established regarding what constituted a valid justification of a generalization, what was considered an efficient generalization, and what was an acceptable use of notation to represent a generalization. The results demonstrate that students readily accept justifications that link to a general relationship within the context of the problem situation. However, some students also appear to accept faulty "proof by example" justifications for generalizations. Students were able to identify the strengths and limitations of recursive and explicit reasoning, though the connections between these types of reasoning often remained tacit in student explanations. (Contains 27 references.) (MM)

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# Developing Middle School Students' Understanding of Recursive and Explicit Reasoning

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Current reform efforts (NCTM, 2000; NRC, 1989) recommend that students receive increased access to algebraic ideas throughout the K-8 curriculum. Through enhancing the K-8 experience, it is believed that students' transition into formal algebra will be improved. To facilitate this transition it is important that students possess a dynamic view of algebraic notation, rather than the view that variables represent a particular unknown to be found (Janvier, 1996). Prior to formal algebra, it is recommended that the curriculum make use of patterning activities in which students generalize numeric situations similar to that in Figure 1. These tasks provide an opportunity to link to student's prior work with arithmetic (Kaput, 1995) and can promote a view of variables as representing a range of values.

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Insert Figure 1 about here

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## Background

Research has found that elementary and middle school students use a wide range of strategies to generalize situations (Kenney, Zawojewski, & Silver, 1998; Stacey & MacGregor, in press; Swafford & Langrall, 2000). Imbedded in their generalization strategies are recursive reasoning (i..e., deriving values in a sequence based on previous values) and explicit reasoning (i..e., using a rule to calculate any term directly). These two types of reasoning are powerful means for constructing generalizations used in advanced mathematics where the interconnections between these two types of reasoning often allow for the development of closed-form representations.

Despite the awareness of the types of algebraic reasoning that students bring regarding algebra, researchers do not provide a clear picture for how to use these two types of reasoning in the classroom. Stacey and MacGregor (in press) recommend that students engage in tasks that focus on explicit reasoning and discourage the use of recursive reasoning. In contrast, Kaput (1993) advocates extending the curriculum beyond the singular focus on explicit rules, allowing iterative mathematics to play a larger role in the curriculum. Building on student reasoning--both recursive and explicit--may be beneficial for students, just as it has been for other mathematical topics

(Hiebert et al., 1997). The question arises whether middle school students are capable of recognizing the strengths, limitations, and connections between these two types of reasoning. Such student recognition has significant implications for instructional and curricular decisions in the middle grades.

Another important issue in the middle school mathematics classroom is whether or not students can justify their generalizations. The elementary school emphasis on finding particular values for numeric situations rather than in expressing general relationships can create difficulties in understanding and justifying general statements. Research has documented that students often attempt to justify general statements using a faulty "proof by examples" argument (Carpenter & Levi, 1999).

A final consideration is students' use of notation as they represent their generalizations. Others (Sfard, 1995; Swafford & Langrall, 2000) have documented that the development of notation appears to follow the historical development of notation, moving from verbal descriptions (e.g., multiply the number of people by 4 and add 2) toward the use of syncopated expressions (e.g., number of people x 4 + 2), to the current use of formal algebraic symbols (e.g., $4n+2$). One of the important aspects of the development of notation is that students need to see the notation as representing a varying quantity, rather than as the static view of an unknown that typifies traditional algebraic instruction (Janvier, 1996).

This study investigated students' generalizations of numeric situations through the use of a whole-class teaching experiment (Cobb, 2000) in which students were provided problem situations in order to:

1. trace students' understanding of recursive and explicit reasoning;
2. describe the sociomathematical norms and classroom practices that were established regarding what constituted a valid justification of a generalization, what was considered an efficient generalization, and what was an acceptable use of notation to represent a generalization.

The classroom environment encouraged students to conjecture and discuss their reasoning in small-group and whole-class settings.

## Conceptual Framework

The conceptual framework used current theory for "teaching experiment" methodology situated in Piaget's constructivist theory of learning (Cobb et al., 1991) with Simon's (1995) Mathematics Teaching Cycle (MTC) as the instructional theory. The MTC included the use of domain-specific knowledge of students' reasoning to develop a hypothetical learning trajectory (HLT) to inform instruction. Therefore, a research base was established prior to the teaching experiment to provide this knowledge. The types of problem situations used included verbal and contextual problem situations (Nathan & Koedinger, 2000; van Reeuwijk, 1995) to stimulate students' informal knowledge and problem-solving strategies. Background on students' strategies for developing generalizations through numeric situations (see Figure 2) was derived from the findings of Healy and Hoyles (1999), Stacey (1989), and Swafford and Langrall (2000). Further guiding the HLT for this study was the view of both recursive and explicit reasoning as valuable means for expressing generalizations.

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Insert Figure 2 about here

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The learning goal for the teaching experiment was to develop students' understanding of the strengths, limitations, and connections between recursive and explicit reasoning when generalizing numeric situations. This goal was subject to modification based on the researcher's understanding of the mathematics and hypotheses regarding students' understanding of the mathematics.

The plan for learning activities included providing problems that could be modeled using explicit and/or recursive reasoning, problems best modeled using explicit reasoning, and problems that the students were most likely to model using recursive reasoning. Five such problems (see Appendix A) were provided during the course of the teaching experiment.

## Method

A class of 25 sixth-grade students from a Midwestern K-8 school participated in a 10-day teaching experiment that gathered quantitative data from students' performance on pre, post, and

retention assessments and qualitative data from the instructional component. The researcher served as the teacher during all of the classroom sessions, and a doctoral student served as the witness. Specific attention was given to four target students (representing a range of ability levels) and their strategies and understandings. The four target students are referred to by the pseudonyms Angela, Adam, Colleen, and Sam.

The teaching experiment followed Cobb's (1999) Developmental Research Cycle, consisting of a developmental phase and a research phase. In the initial developmental phase, a research base was constructed to guide instructional decisions allowing for the construction of a HLT (Simon, 1995) and the selection of classroom tasks. The tasks were implemented in the research phase during which results were gathered from classroom instruction and target student interviews. The information that was collected in the research phase was utilized in the developmental phase to guide further instructional decisions.

Instruction was guided by the desire to establish classroom practices related to three sociomathematical norms: (a) What constitutes a valid justification of a generalization?; (b) What constitutes an efficient generalization?; and (c) What constitutes an acceptable mathematical representation? Small-group and whole-class discussions were designed to elicit discussion about the acceptable modes of operation in the classroom related to these three sociomathematical norms.

The typical instructional format was to read and discuss the important characteristics of a problem situation, followed by individual thinking time to encourage each student to develop his or her own strategy for generalizing the situation. After the individual time, small-group and whole-class discussions were used to establish "taken as shared" (Cobb, 2000) means of operating in these situations. All of the class sessions took place in a computer lab in which students were allowed to use computer spreadsheet technology to test their generalizations. Two introductory sessions on how to enter formulas into a spreadsheet and organize information were provided prior to the introduction of generalizing numeric situations.

The study utilized a mixed-method design (Tashakkori & Teddlie, 1998), gathering and analyzing qualitative and quantitative data. Pre, post, and retention assessments (which required

students to generalize three numeric situations) served as data for examining changes in student understanding. Further information was provided through observation notes and five individual target student interviews that were conducted following alternate class sessions (see Figure 3).

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Insert Figure 3 about here

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Videotape data, students' written work, and computer spreadsheet files served as further evidence of students' reasonings. The target students were videotaped during small-group discussions, and the entire class was videotaped during whole-class discussions. All of the students' written responses to the generalization activities were gathered. The computer spreadsheet files were copied following each class session, providing a snapshot of the students' reasonings at the end of the session. In addition, at the conclusion of each instructional session, discussions between the researcher and the witness regarding their observations and suggestions for modifications to the HLT were audiotaped. The variety of data sources allowed for triangulation of data and different perspectives and interpretations of the data (Eisenhart, 1988), providing a broad picture of student development.

Using a researcher-constructed rubric, scores were assigned to the assessments. All of the assessments were scored independently by the researcher and by another mathematics educator who was trained to score the assessments (inter-rater reliability of 0.93). These data were analyzed using a repeated-measures Analysis of Variance (ANOVA) to determine whether a significant change occurred in the students' performance.

The qualitative data from the interviews and instructional sessions were analyzed using a data reduction approach (Miles & Huberman, 1994) to discern key patterns in student reasoning and changes in reasoning. All transcripts from the videotapes and interviews were coded using an initial list of codes that was extended, as needed, during the coding process. These codes described strategies used by both the target and non-target students for attempting to develop generalizations (e.g., recursive, whole-object, etc.), conjectures voiced by the target students, the influence of

technology, the symbolic representations that the target and non-target students generated, and the justifications provided for students' generalizations.

Chronological cross-case displays (Miles & Huberman, 1994) were constructed for each task and interview session that occurred during the teaching experiment. A target student-by-task meta-matrix was then constructed to discern changes in strategies over time. This matrix allowed for comparisons across cases and across tasks. Comparisons were made regarding students' strategies for each task provided and individual strategy changes during the teaching experiment.

## Results

The results of a repeated-measures ANOVA showed a significant increase in student ability to construct and explain their generalizations between the pre and post assessments [$F(1,24) = 6.61$, $p < .05$] and the pre and retention assessment [$F(1,24) = 7.30$, $p < .05$]. However, the average scores for the pre, post, and retention assessments were quite low, indicating the continued difficulty that students had generalizing situations. Qualitative analysis of the assessments demonstrated that student use of counting and the whole-object strategies decreased following the teaching experiment.

### Student Use of Recursive and Explicit Reasoning

The results demonstrate that many students successfully used recursive and explicit reasoning to construct their generalizations, though some students also exhibited difficulty in constructing generalizations. Table 1 provides the types of generalizations the students generated for the five problem situations. The generalizations are categorized by whether the generalization strategy properly modeled the problem situation (appropriate) or contained a mismatch between the problem situation and the generalization strategy (inappropriate). The generalizations are further categorized by those strategies that used recursive reasoning and those strategies that used explicit reasoning.

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Insert Table 1 about here

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Table 1 shows how students' strategies varied depending on the problem situation. The problem that provided a direct connection to recursive reasoning, the *Theater Seats* problem, generated the greatest use of this strategy. The problem for which an explicit strategy is most accessible, the *Pizza Sharing* task, generated only explicit rules. The *Streets and Lampposts* problem was the most difficult situation for students, which only nine students were able to generalize. For this situation, the only student who used what was classified as an explicit strategy, was the cumbersome rule $L = 1 + 2 + 3 + \ldots + (n - 1)$ where n is the number of streets and L is the number of lampposts. (Note that the use of notation here is my notation, not that of the students. The students often used syncopated or verbal expressions to describe their generalizations. This is further described in a later section.)

Students were observed using all of the generalization strategies identified in the initial framework (see Figure 2) as well as another explicit generalization strategy, the rate-adjust strategy, which had not been previously identified in the research literature. This strategy involved reasoning abstractly about the relationship among the dependent values in an arithmetic sequence. For this strategy a student first determined the rate of change ($m$) between consecutive terms and multiplied $m$ by a particular value of the independent variable (usually the initial value in the domain of the independent variable), adding or subtracting an adjustment value ($a$) to provide the corresponding value for the dependent variable. This generated the expression $mn + a$ where $n$ was the independent variable. This expression was then used to find other values of the dependent variable. Only one student was observed using this strategy over the course of the study. He was successful with modeling linear problem situations, but was unable to construct generalizations for the non-linear (*Pizza Sharing* and *Streets and Lampposts*) situations.

Aspects Inhibiting Students' Constructions of Generalizations

The factors inhibiting students' ability to develop generalizations were students' limited understanding of the meaning of operations and the relationship among operations, their focus on particular values rather than general relationships, their difficulty in formalizing mental calculations,

and their calculational rather than conceptual orientation. These factors relegated some students to use the guess-and-check strategy and led to difficulties in justifying generalizations.

For example, during the *Phone Cost* problem Angela was focused on finding particular values rather than using a general relationship do develop a general rule. Below is an excerpt from the interview in which she describes how she attempted to find a rule for the cost of a phone call that was longer than 5 minutes in length.

Interviewer: Yesterday, you did a really interesting thing with the phone problem. Can you describe what you were doing?

Angela: I did A6 (the referent cell) times 9.5 and I got 57.

Interviewer: Ok, the 9.5, what information did that provide?

Angela: It was close to 56. I couldn't get right on 56 so I tried decimals, except that it didn't quite work. I got 57 and that was as close as I could get.

Angela attempted to find a constant that could be multiplied by 6 that would result in the correct cost for a 6-minute phone call (56¢). She began by multiplying 6 by 9.3, 9.4, 9.5, and 9.6 using the computer spreadsheet. When 9.5 resulted in the value that was closest to 56, she discontinued this process. She appeared to be so focused on particular values that her ability to construct a generalization was inhibited, and she seemed unaware that the rule she created would provide little help for determining the cost of phone calls that were longer than 6 minutes.

When students were able to find correct values for particular instances, they exhibited difficulty translating their mental calculations into formal generalizations. For example, during her work on the *Phone Cost* problem Colleen was able to find the cost for a phone call that took six minutes or more. Her written strategy was to take the number of minutes and subtract 5 from this value mentally, then she would multiply the result by 6 and add 50. When she attempted to translate her written strategy into a general rule on the computer spreadsheet, Colleen had difficulty determining with the mental computation of subtracting 5. She began by using the rule $6n + 5(10)$, but realized that this resulted in a cost that was too high. She tried subtracting 5 at the end of her

rule and concluded that this, too, was incorrect. Her inability to connect to what the particular components of her formula represented inhibited her progress toward an appropriate generalization.

A final inhibiting factor was some students' focus on finding a generalization that provided the correct values rather than understanding the process used to construct a generalization. When the target students discussed in their small groups, a student often provided his or her rule without justifying it. Attempts by the researchers to encourage students to focus on the process in the small groups were generally ignored by the target students. Instead the target students used their computer spreadsheet software to test rules for various values.

Sociomathematical Norms and Classroom Practices

Five classroom practices (see Figure 4) emerged that corresponded to the three identified sociomathematical norms: what constituted a valid justification, what constituted an efficient generalization, and what was an acceptable mathematical representation. The classroom practice expectation emerged that providing particular examples was not a valid means of justifying a generalization. The students also established that connecting a generalization to the invariant conditions that existed in the context of the situation was an appropriate means of justification.

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Insert Figure 4 about here

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Regarding the efficiency of generalizations, students demonstrated a strong awareness of the limitations for recursive reasoning when generating large values of the independent variable. To a lesser extent, the students recognized that recursive reasoning was an efficient means for determining consecutive values of the dependent variable and that recursive formulas were sometimes easier to construct than explicit formulas.

When representing generalizations, the use of verbal descriptions (e.g., four times the number of cubes plus two), syncopated expressions (e.g., 4 x number of cubes + 2), and spreadsheet formulas (e.g., =4*A2+2) emerged as acceptable means of operating in the classroom.

Overall, the students appeared to prefer the use of verbal descriptions, perhaps due to their concern for clarity of their representations rather than conciseness.

Valid justifications. Throughout the teaching experiment, both the witness and I encouraged the students to connect their generalizations back to the context of the problem situation in both small-group and whole-class settings. The students readily accepted justifications that established a general relationship that existed in the problem situation. Despite this, students often used specific examples to verify their generalizations and a discussion of the validity of justification through the use of examples arose only once in the whole-class setting. In addition to these two types of justifications, one of the target students, Sam, also provided an informal proof by induction to defend his use of the rate-adjust strategy, though this justification was provided only during individual interviews and not in the whole-class setting.

An example of an explanation that was accepted by the class is provided by Sam as he explains his explicit rule $S = 4n + 2$ to the class (where $S$ is the number of stickers and $n$ is the length of the rod) for the *Cube Sticker* problem.

Teacher: I have six cubes here. Can you explain why you are multiplying by four?

Sam: Because (there are) 6 (sides) here, 6 here, 6 here, and 6 there (pointing to the top, left, right, and bottom faces of the rod).

Teacher: Ok, why are you adding two?

Sam: Because of these two (sides) [pointing to the two ends of the rod].

Sam described why six stickers were necessary for four faces of the rod and two stickers were required for the two ends. Further discussion in class showed that the students understood how this relationship could be extended to find the number of stickers for a rod of any length.

A discussion occurred in the whole-class setting for the *Theater Seats* problem regarding whether a rule could be justified based on the knowledge that it provided the correct values for a finite number of cases. During the discussion, some students expressed concern when a student explained that he had derived his formula through a guess-and-check strategy and determined that

his generalization was valid based on testing particular values. As I solicited other students' opinions of the validity of this argument, the following discussion ensued.

Teacher: (Student 1), please explain your rule.

Student 1: How I got it is I multiplied by 4 and I added 2. It was like a guess and check.

Teacher: Ok, so you guessed and it worked for the first [case].

Student 1: Yeah, and I checked it for others and it worked.

Teacher: And it worked for the second [example] and it worked for the third [example]. Is it always going to work?

[I then asked for a show of hands for students who agreed with Student 1's reasoning and those who did not. The class was divided approximately evenly between these two groups.]

Teacher: [Student 2], what do you think?

Student 2: Well, I don't really understand what he is doing. I don't think that it will work every time. It may work for a few, but it may not work for some numbers.

Student 2 questioned whether the rule would continue to model the situation appropriately based on its correctness for a few particular instances. There was little time left for further discussion about the validity of this claim, though in the discussions that followed, no student provided a "proof by example" argument in the whole-class setting. Despite this, the target students continued to use this type of justification during the individual student interviews.

The third type of justification was given by Sam to validate his rate-adjust method. His justification is an example of an informal proof by induction. An excerpt from his explanation for his $6n + 20$ rule for the *Phone Cost* problem is below.

Teacher: Explain your formula Sam.

Sam: Well, I just did times 6 cents because it's six cents every minute. And when I put in 6 I said 6 times 6 equals 36. Then I just added 20 to get 56 because I knew 50 + 6 was 56. And then I wanted to make sure that it didn't work for only six. So I did seven, and it went up six, and it's supposed to do that. And then I did 8, and it went up six, and it's supposed to do that.

Sam stated that the cost of a phone call was correct for 6-minute phone call, and that the cost increased by 6¢ as the length of the phone call increased by 1 minute. It was not completely clear that Sam understood why his rule increased by 6, but he consistently used this information to construct generalizations for linear situations.

Efficient Generalizations. During the teaching experiment the target students and the other students in the class occasionally referred to the strengths, limitations, and connections between recursive and explicit reasoning. This lead to the accepted norm that explicit rules were simpler to use for larger values of the independent variable. In addition, two emerging classroom practices involved students noting the simplicity of using recursive rules to find consecutive values of the dependent variable, and that recursive rules were often simpler to construct than explicit rules.

In order to encourage students to examine the strengths and weaknesses of the strategies that were presented in the whole-class discussions the students were asked by the teacher/researcher, "Which strategy is preferable?" For the *Cube Sticker*, *Theater Seats*, and *Phone Cost* problems the students expressed a preference for the use of explicit formulas, especially those formulas that used the fewest number of operations. The only problem situation where the students expressed partiality for a recursive rule was the *Streets and Lampposts* problem, due to the laborious nature of the explicit formula that was presented.

The connections that students made between recursive and explicit reasoning were more elusive. For most of the situations, connecting these two types of reasoning involved understanding that repeated addition could be expressed as multiplication. Rarely did any student make direct reference to this relationship, apparently assuming that everyone understood this connection.

Strengths and limitations. Identification of the limitations of recursive reasoning occurred repeatedly during the *Theater Seats* problem. Because two of the target students, Adam and Colleen, had constructed an explicit formula for the *Cube Sticker* problem, they developed an understanding of some of the advantages of using an explicit formula. They used this perspective to examine the functionality of using a recursive formula. A discussion arose about the efficiency of

using a recursive rule as Angela explained to Colleen how to build her rule ($u_n = u_{n-1} + 4$, $u_1 = 6$. Note: notation mine) on the spreadsheet. An excerpt of this discussion is given below.

Angela: (To find the number of) seats this is what you do. You (enter) six and then you press return, then you do equals. And then you (select the referent cell containing the six) and then you do plus 4. You keep on adding 4 each time right? Hit return and then you click here and copy. Then you do edit . . .

Colleen: Yeah, but then how do you do the really high, high numbers?

Angela: You just do the same thing. You do copy (and) paste.

Colleen: What do you do about (row) 378? You have to know how much 377 is..

Witness: Colleen, what were you saying? To know (the number of seats in row) number 378 . . .

Colleen: You gotta know (the number of seats in row) 377.

Larger values provoked Colleen to question Angela's use of a recursive strategy. She recognized that the recursive rule required computing the number of seats in the previous row. Angela demonstrated that copying and pasting her rule down the column of her spreadsheet allowed for finding the number of seats in the 378th row, but Colleen questioned how efficient this was. This discussion was representative of those that occurred during whole-class discussions for the *Theater Seats* and the *Phone Cost* problems. However, during discussion of the *Phone Cost* problem a student mentioned that explicit formulas were somewhat more difficult to construct than recursive formulas.

The preference for the use of a recursive formula was exhibited when the target students were asked to compute consecutive values of the dependent variable for the *Cube Sticker* and the *Theater Seats* problems. Both Adam and Sam stated that they used a rule to find the next output by building on the previous term in the sequence, but they recognized that an explicit formula was more efficient for larger input values. Adam's explanation of this in relation to the *Theater Seats* problem is given below.

Adam: You just . . . . add. For the first couple (of rows) I like the "add four" [rule] ($u_n = u_{n-1} + 4$), but for the bigger numbers, I like mine ($4n + 2$) cause it was easier and quicker.

Both Adam and Sam used a recursive strategy to create a table of values, but no mention was made during the whole-class discussion that the recursive rule provided a simpler method in this situation. This may have been due to the focus on large values for the problem sheets. The *Cube Sticker* problem was the only situation in which students were directed to find the outputs for consecutive input values.

For the *Streets and Lampposts* problem the students changed their view on the use of a recursive rule [$u_n = u_{n-1} + (n\text{-}1)$, $u_1 = 0$]. In the excerpt below, Student 1 described his recursive strategy, and another student explained why Student 1's recursive rule was preferable over the explicit rule [$1 + 2 + 3 + \ldots + (n - 1)$].

Teacher: What do you think about these two ways so far? Whose do you prefer?

Student 1: Well, I'll tell you about the formula that I used to get what I got is I added. I minused one from streets like (the other student did for his explicit strategy), but then I just added the number above it because that is the number that all of the others had already used. So for (to find the number of lampposts for) 8 streets I'd add 21 (the total number of lampposts necessary for 7 streets) minus . . . I'd minus one from 8, which is 7 and then I'd add 21, which is above it.

Teacher: Then you copied and pasted that formula down. What do you (the class) think of that strategy? Is that the best way to do it? Is this (the explicit rule) better?

Student 2: I don't know but (Student 1)'s works pretty quickly.

Teacher: It works pretty quickly, we could find the number of lampposts if there were 13 streets. What about (the explicit) strategy? Does it work (just as) well?

Student 3: It was a little harder because you have to add 12, 11, 10, 9, . . .

Teacher: A little harder. What if I wanted to [find the number of lampposts for] 100 streets]?

Student 4: 99 + 98 + . . .

The difficulty in translating the explicit rule into the computer spreadsheet appeared to deter students from choosing the explicit strategy provided. The students were unable to construct the explicit rule L = [S (S-1)]/2 (where L is the number of lampposts and S is the number of streets) during the time allotted in class. Because the spreadsheet allowed the students to add to the previous total of the sum expressed in the explicit rule it provided a distinct advantage over the explicit strategy.

Recognition of interrelationships between recursive and explicit reasoning. Connecting the recursive relationship to an explicit rule appeared to be tacit knowledge for some students, whereas others appeared to lack an understanding that this information was useful or understand how it could be used. For many of the explanations that were provided in the whole-class discussions, the connections were given only a quick reference. For the four target students, Adam and Angela rarely appeared to use the connection, though Colleen and Sam often used the connection to create their explicit rules.

An example of how Sam used the recursive relationship to construct another explicit formula occurred in the first interview following the introduction to the first generalization situation, the *Cube Sticker* problem. He initially described his explicit rule ($4n + 2$) and how he used his rate-adjust strategy, but then he moved to examining the recursive relationship when he focused on the change from rods of consecutive length. Using this relationship, he built a new explicit rule for finding the number of stickers on a length-20 rod. The excerpt below describes his reasoning.

Sam: I did the exact same thing with nine.

Interviewer: Ok, so for nine you did what?

Sam: I [did] 9 times 4 is 36 and 36 plus 2 is 38.

Interviewer: Ok.

Sam: And then I went down to eight, and I did the same thing. And then when I got to seven I realized that all it is is going down by four. So 42, 38, 34, 30, and then I just filled in the rest.

[Sam then was asked to explain why the recursive relationship occurred in this situation. Following this discussion the dialogue continued below.]

Interviewer: Can you tell me how you would find the number of stickers on a rod of length 20?

Sam: I [know a rod of length 10 has] 42 so I would just add 40 on. That would be 82, and I know that it's not 84 because you're not making another one of these. You're keeping these (ends). All you're doing is counting four more each time.

Interviewer: Ok, tell me that again.

Sam: Then I'm going to add another 10 without the sides (ends), and I get 40. And then 42 plus 40 equals 82.

Sam moved quickly from the recursive relationship to generating his new rule [$42 + 4(n\text{-}10)$]. He was able to establish a connection with the number of stickers needed for each cube, 4, to multiplying by the number of extra cubes necessary for this situation.

Such a connection between the recursive relationship of adding 4 each time and multiplying 4 by a particular value appeared in the discussion of the *Cube Sticker* problem of the explicit rule $4(n\text{-}2) + 10$. The student describing this rule appeared to see no connection between her recursive rule and an explicit rule until I encouraged extending her recursive rule to an explicit rule.

Teacher: Let's look at (Student 1)'s strategy here for just a second. (Student 1), tell us what you'd do if there were, let's say 23 cubes.

Student 1: If there were 23 I would count six for the first one, but then when I added the second one I would get five.

Teacher: So there are five. There are five stickers on one of the cubes.

Student 1: Right.

Teacher: On one of the ends.

Student 1: And then I would just keep adding four.

Teacher: Why are you adding these fours?

Student 1: Because there are only four areas showing. Because there is another cube blocking this area (the end of the cube).

Teacher: So you add a bunch of fours for cubes in between. You keep adding a bunch of fours, and then you're done?

Student 1: No, and then on the last one you add five also.

Teacher: Can anybody tell me how many fours she would add there? Student 2?

Student 2: I think there are 21 because there is one five for the first cube and one five for the last so take away two cubes and you'll have it.

Teacher: So can somebody tell me an easier way to do this than to add a bunch of fours? I mean, I can add a bunch of fours. But is there an easier way to do it?

Student 3: Twenty-one times four.

Student 1 appeared unable to extend her recursive rule into an explicit rule without some assistance from the teacher and her classmates. This conversation is representative of others that occurred at various times during the teaching experiment, demonstrating that many students did not exhibit an understanding of the manner in which a recursive rule could be used to construct an explicit formula for the *Cube Sticker*, *Theater Seats*, and the *Phone Cost* problems.

Implicit connections between the recursive and explicit rules occurred regularly in the whole-class discussions. Below is an example of a student describing her strategy for the *Phone Cost* problem to find the cost of a call that was more than five minutes in length.

Student: What I did was I took the number of minutes the phone call was going to be, and I subtracted five from it because that was the first five minutes . . . The first five minutes that cost more, cost 50¢. Then I took the number of minutes that I got after I subtracted five and timesed it by six because every

> minute after that was 6¢. And then I added 50, which was the (cost) of the first five minutes that I subtracted the five from the minutes (at) first.

In her explanation the student assumed that everyone understood the connection between the recursive relationship of adding 10¢ or 6¢ per minute and the explicit rule that she described. She alludes to a connection between the cost per minute and multiplying by 6, but makes no direct reference to how repeatedly adding connected to multiplying by 6 for her rule.

Acceptable representations. The final sociomathematical norm that generated particular classroom practices involved students' representations of generalizations. The classroom practice emerged during the study that the use of verbal descriptions, syncopated expressions, and spreadsheet formulas were acceptable means to describe a generalization. During the teaching experiment the majority of the students wrote verbal descriptions (e.g., "I started with six stickers and added 4 each time for each extra cube that was added to the rod") to express their explicit or recursive rules. Initially, students used verbal descriptions, syncopated expressions (e.g., number of cubes x 4 + 2), spreadsheet formulas (e.g., =B4*4 + 2), partial calculations (x 4 + 2), and blanks to represent variables ( __ x 4 + 2). Despite the preference for verbal descriptions, there was a gradual progression toward increased use of symbolic representations over the course of the study (see Table 2). During the class discussion on representing formulas the students explained why they preferred the use of verbal descriptions to the use of any type of symbolic representation. Many students expressed concern with clearly communicating the meaning of their generalizations and were reluctant to use symbols that were not widely understood.

The students' representations moved from verbal descriptions toward an increased use of syncopated expressions and spreadsheet formulas. Table 8 displays the number of students who constructed each of the various representations for each task given in the teaching experiment. Ignoring the *Streets and Lampposts* situation, which was difficult for students to generalize, the table shows a gradual shift in the use of verbal descriptions, syncopated expressions, and spreadsheet formulas. Student use of non-standard symbolic representations occurred only when students wrote explicit formulas. It should be noted that a brief whole-class discussion on

representing formulas occurred during the *Pizza Sharing* problem. Following this discussion no students provided partial calculations when representing their generalizations. The details of the discussion follow.

During class session #7 (following a discussion of the generalization strategies for the *Pizza Sharing* problem) the teacher/researcher asked the students to discuss how to write their formulas for this situation. No formal discussion of representing generalizations occurred prior to this problem. Instead, students were encouraged to represent their generalizations in any manner that they desired. It was hypothesized that a discussion about possible representations would be appropriate during this class session because students had been provided opportunities to generalize various problem situations.

All students discussed in their groups how to represent their formulas. Below is an excerpt from the target students' discussion.

| | |
|---|---|
| Teacher: | In your group right now I want to make sure that everyone has a formula down there (on their worksheet). (You may write your representation) however you want to write your formula. Write a formula that will allow you to find the number of pieces of pizza for any number of students. So talk about that in your groups. I will give you a couple of minutes to do that in your group. |
| Colleen: | Ok, write a formula that calculates the number of pizza for any number of students. Ours does that. |
| Angela: | No matter what number of students you have . . . |
| Colleen: | We can write in ours. |
| Adam: | What was our formula? |
| Colleen: | It was equals 72 divided by whatever. |
| Adam: | Whatever your thing (referent cell) was, A2, A3, A4. |
| Sam: | How many (or the number of) students changes. |

Colleen: Yeah, you could do like 100 (students) and do it (find the number of pieces of pizza per person.)

The target students' discussion involved the use of a spreadsheet representation for their generalization. Colleen and Sam suggested that the number of students in the situation varied in this context, and Adam reinforced the idea that the referent cell label did not matter in this situation. Angela noted the general nature of the formula, that it would calculate the number of slices of pizza for any number of students. Every member of the target group wrote a formula using spreadsheet language (e.g., = 72/A4) except for Angela who wrote "= 72/4" and explained what the 72 and 4 represented in the problem situation.

During the whole-class discussion the students expressed concern with the clarity of the representations. A verbal description and a syncopated expression were provided by students: "total amount of pieces in 2 pizzas (72) divided by the number of students," and "72/number of students." Two students suggested that both rules provided the same information, but that the shorter formula was preferred. The discussion then moved toward a question of the use of the symbol "/" for division. A third student suggested that the standard "÷" symbol should be used, and another student indicated that the formula should better describe the result that the formula calculated. The phrase "to get the number of pieces each person would get" was attached to the beginning of the expression "72/number of students." Several students voiced the opinion that being clear on what the formula represented was of greater concern than the concise use of notation.

The results demonstrated the students' preference for verbal descriptions of their generalizations. However, there was some movement of students toward the use of symbolic representations. Students used spreadsheet notation (A4), words (number of students), and blanks as non-standard symbolic notation to represent variables. Such representations do suggest that students viewed their notation dynamically. That is, they saw their notation as representing a range of possible input values for the independent variable.

## Discussion

Previous introductory algebra studies have examined the generalization strategies and notation that emerge as students generalize numeric situations. This study builds on these prior studies to examine how students justify their generalizations and students' awareness of the strengths, limitations, and connections of their use of recursive and explicit reasoning. In addition, this study traced the development of students' notations over an extended period of time.

The results demonstrate that students readily accept justifications that link to a general relationship within the context of the problem situation. However, some students also appear to accept faulty "proof by example" justifications for generalizations. This acceptance of "proof by examples" is well documented (Hoyles, 1997; Sowder & Harel, 1998) and has important implications for the current algebra curriculum. Many current "problems" provided in curricular material appear to promote this kind of reasoning through their use of patterning activities. Problems 1 and 2 (see Figure 5) are examples of patterning activities that are difficult to justify because a unique relationship cannot be discerned in these situations. Rather than using ambiguous patterning situations, it may be beneficial for students to generalize situations such as those used in this study, in which students' generalizations can be properly justified.

Students were able to identify the strengths and limitations of recursive and explicit reasoning, though the connections between these types of reasoning often remained tacit in student explanations. The ability of students to see the strengths and limitations of recursive reasoning suggests that students could naturally move toward the use of explicit reasoning, rather than continue to focus on recursive rules as has been documented by Stacey and MacGregor (in press). In a manner similar to the student progression in strategy for Cognitively Guided Instruction (Carpenter, Franke, Jacobs, Fennema, & Empson, 1997), students may progress if the are encouraged to use their initial strategies, see the connection between these two types of reasoning, and then are encouraged to build explicit rules from recursive relationships.

The students in this study appeared to follow the historical perspective (Sfard, 1995) for the development of notation: beginning with verbal descriptions, followed by syncopated

representations, and concluding with the use of symbolic forms. Rather than use mathematical symbols, most students preferred the use of verbal descriptions for both the quantities and the mathematical operations of a generalization. As such, the use of these representations appeared to provide a dynamic view of the variable quantities in these situations, portraying the quantities as representing a range of values. The use of verbal representations tended to eliminate misconceptions (e.g., the use of variable as objects rather than as numbers) that have been identified in students' use of formal symbols (MacGregor & Stacey, 1997), and instead focused on the meaning of any representation in relation to the context (Pimm, 1995). Such use of verbal descriptions could provide a meaningful link to the introduction of formal mathematical symbols, building the view of a variable as representing a range of values.

Future research needs to examine how student barriers to the development of generalizations can be overcome, the types of situations that help students see connections between the use of recursive and explicit reasoning, and how formal algebraic notation can be introduced. Generalization is at the heart of mathematical activity and needs to receive increased emphasis in elementary and middle school classrooms.

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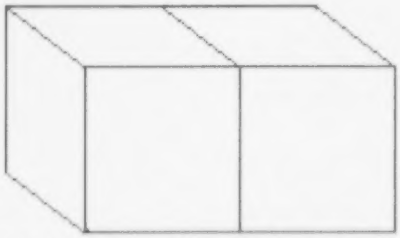
**Cube Sticker (Flexible use of recursive and explicit reasoning)**



A company makes colored rods by joining cubes in a row and using a sticker machine to place "smiley" stickers on the rods. The machine places exactly one sticker on each exposed face of each cube. Every exposed face of each cube has to have a sticker, so the above length-two rod would need 10 stickers.

1. How many stickers would you need for rods of lengths 1 to 10? Explain how you determined these values.
2. How many stickers would you need for a rod of length 20? Of length 50? Explain how you determined these values.
3. Suppose that a particular rod needed 150 stickers. What is the length of this rod? Explain how you determined this.
4. Explain how you could find the number of stickers needed for a rod of any length. Write a formula that you could use to determine this.

Figure 1. The Cube Sticker Problem.

| Strategy | Description |
|---|---|
| **Non-Explicit** | |
| Counting | Drawing a picture or constructing a model to represent the situation to count the desired attribute. |
| Recursion | Building on the previous term or terms in the sequence to determine the next term. |
| **Explicit** | |
| Whole-object | Using a portion as unit to construct a larger unit by multiplying (e.g., 3 apples cost $8 so 9 apples cost $24.) There may or may not be an adjustment for over or undercounting. |
| Guess-and-check | Guessing a rule without regard to why this rule may work. |
| Contextual | Constructing a rule based on information provided in the situation by connecting to a counting technique. |

Figure 2. Student Generalization Strategies

| Class Session | Activities | Target Student Interviews |
|---|---|---|
| #1 | Introductory Spreadsheet Problem | |
| #2 | Introductory Spreadsheet Problem | |
| #3 | Cube Sticker Problem (F) | X |
| #4 | Cube Sticker Problem (F) | |
| #5 | Theater Seats Problem (F) | X |
| #6 | Theater Seats Problem (F) | |
| #7 | Pizza Sharing Problem (E) | X |
| #8 | Pizza Sharing Problem (E)<br>Phone Cost Problem (F) | |
| #9 | Phone Cost Problem (F) | X |
| #10 | Streets and Lampposts Problem (R) | |

Figure 3. Tasks and Interview Schedule for the Teaching Experiment.

Note. F implies the problem allowed for the flexible use of recursive and explicit reasoning, E implies the problem was explicit-preferred, and R implies the problem was recursive-preferred.

| Sociomathematical Norms | Classroom Practices |
|---|---|
| What constitutes a valid justification? | Examples do not constitute justification of a generalization |
| | Relating to the context (in general terms) is a valid means of justification |
| What constitutes an efficient generalization? | Explicit rules are simplest to use when calculating large values of the independent variable |
| | Deriving recursive rules can be simpler, and these rules are easier to use when finding consecutive values of the dependent variable |
| What constitutes an acceptable mathematical representation? | Acceptable means of representation are verbal descriptions, syncopated expressions, and spreadsheet formulas |

Figure 4. Sociomathematical Norms and their Corresponding Classroom Practices

**Problem 1**

Examine the pattern below:

2, 4, 6, 8, 10, . . .

Find the next 3 numbers in this pattern.

**Problem 2**

The pattern continues.

How many squares are there in the 6th figure in this sequence?

Figure 5. Ambiguous Patterning Problems

| | Appropriate Generalizations | | Inappropriate Generalizations | | |
|---|---|---|---|---|---|
| Problem Situation | Recursive | Explicit | Whole-Object (overcounting) | Other | No Generalization Apparent |
| Cube Sticker | 6 | 14 | 3 | 1 | 4 |
| Theater Seats | 12 | 14 | 0 | 2 | 0 |
| Pizza Sharing | 0 | 22 | 0 | 2 | 4 |
| Phone Cost | 6 | 15 | 0 | 0 | 4 |
| Streets and Lampposts | 8 | 1 | 0 | 2 | 14 |

Table 1. Generalization Strategies Used During the Teaching Experiment

Note. The total number of strategies is more than 25 due to the use of multiple strategies by some students.

| | Task | | | | |
|---|---|---|---|---|---|
| Representation | Cube Sticker | Theater Seats | Pizza Sharing | Phone Cost | Streets and Lampposts |
| Verbal | | | | | |
| Description | 12 | 9 | 12 | 9 | 0 |
| Syncopated | | | | | |
| Expression | 2 | 4 | 4 | 5 | 1 |
| Spreadsheet | | | | | |
| Formula | 2 | 1 | 4 | 4 | 0 |
| Partial | | | | | |
| Calculation | 2 | 3 | 0 | 0 | 0 |
| Blank as | | | | | |
| Variable | 1 | 0 | 0 | 0 | 0 |

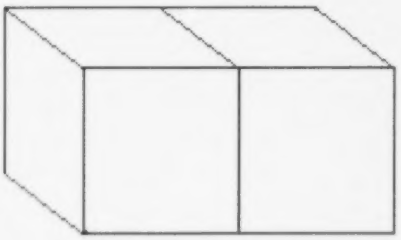
Table 2. Student Representations by Task during the Teaching Experiment

# APPENDIX A

## GENERALIZATION PROBLEMS CONSTRUCTED FOR USE IN THE TEACHING EXPERIMENT

**Cube Sticker (Flexible use of recursive and explicit reasoning)**

A company makes colored rods by joining cubes in a row and using a sticker machine to place "smiley" stickers on the rods. The machine places exactly one sticker on each exposed face of each cube. Every exposed face of each cube has to have a sticker, so this length two rod would need 10 stickers.



1. How many stickers would you need for rods of lengths 1 to 10? Explain how you determined these values.

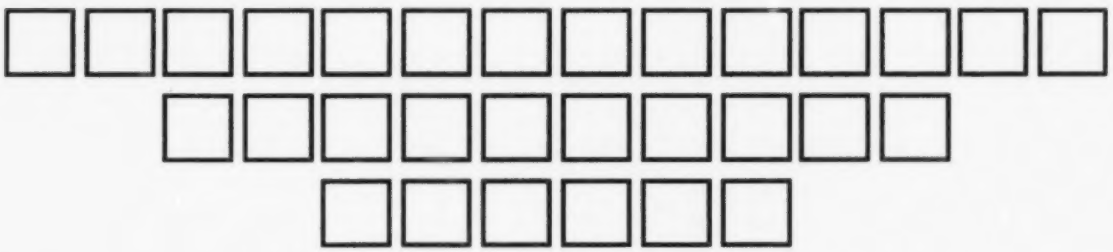
2. How many stickers would you need for a rod of length 20? Of length 50? Explain how you determined these values.

3. Suppose that a particular rod needed 150 stickers. What is the length of this rod? Explain how you determined this.

4. Explain how you could find the number of stickers needed for a rod of any length. Write a formula that you could use to determine this.

**Theater Seats Problem (Flexible use of recursive and explicit reasoning)**

In a theater there are 6 seats in the first row. Each row after the first row contains 4 more seats than the row before it. Below is a diagram of the first three rows in the theater.



1. How many seats are there in the $7^{th}$ row of the theater? In the $12^{th}$ row? Explain how you determined these.

2. How many seats are in the $104^{th}$ row of the theater? Explain how you determined this.

3. In what row would there be 378 seats? Explain how you determined this.

4. Explain how you would determine the number of seats in any row. Write a formula that would allow you to calculate the number of seats in any row. Explain your formula.

**Phone Cost (Flexible use of recursive and explicit reasoning)**
The Jog Phone Company is currently offering a calling plan that charges 10¢ per minute for the first 5 minutes for any phone call. Any additional minutes cost only 6¢ per minute.

1. How much would a 3-minute phone call cost? An 8-minute phone call? A 15-minute phone call? Explain how you determined these.

2. How much would a 123-minute phone call cost? Explain how you determined this.

3. Suppose that a certain phone call (using the Jog Company plan) costs $4.64. How many minutes long was the call? Explain how you determined this.

4. Explain how you would determine the cost for any number of minutes under this plan. Write a formula to explain how you would determine this cost.

**Pizza Sharing (Explicit Reasoning Preferred)**
The LargeFoot Pizza Company sells a large pizza that contains 36 pieces. Your class has purchased two of these pizzas for an upcoming class party. Everyone has agreed that each person will receive the same amount of pizza.

1. Suppose only 4 students attend the class party, how many pieces of pizza would each student receive? What if 9 students attend the class party? Explain how you determined these.

2. Suppose that 30 students attend the class partly, how many pieces of pizza would each student receive. Explain how you determined this.

3. At the party all students receive one and a half slices of pizza. How many students attended the party? Explain how you determined this.

4. Write a formula that would allow you to find the number of pieces of pizza for any number of students. Explain your number sentence.

**Streets and Lampposts (Recursive reasoning preferred)**
In Crazytown every street intersects every other street exactly once. The streets are not necessarily straight. Only two streets meet at any one intersection. The city wishes to erect a lamppost at every intersection.

1. How many lampposts would be needed if there was1 street in Crazytown? 2 streets? 3 streets? 4 streets? 5 streets? 14 streets? Explain how you determined these.

2. How many lampposts would be needed if there were 42 streets in Crazytown? Explain how you determined this.

3. Suppose that it is determined that 4186 lampposts are needed to place a lamppost at each intersection. How many streets must there be in Crazytown?

4. Write a formula that would allow you to find the number of lampposts for any number of streets. Explain your formula.